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ON SOME CASES WHEN THE QUINTIC IS SOLVABLE BY ELEMENTARY METHODS.

By A. C. BURNHAM, University of Illinois.

Abel has shown that the general equation of the fifth degree is not solvable by the ordinary method of extraction of roots. Abelian equations and cyclical equations in which the *roots* satisfy a certain condition can be so solved. It would be of value to have the necessary and sufficient condition which must be satisfied by the *coefficients*, instead of by the roots, in order that the given equation be an Abelian, or a cyclical, or, in fact, be solvable by root extraction, so that one could immediately tell, given an equation, whether it were solvable. I believe no one has as yet deduced such a condition for the general quintic.

It is the object of this paper to give, not the general condition indeed, but one which covers many special cases, in the hope that it may prove of interest or of value in further study of the subject.

Let the general quintic be

$$f(x)=x^5+a_1x^4+a_2x^3+a_3x^2+a_4x+a_5=0 \quad \ldots\ldots\ldots\ldots (I),$$

and let it be divisible without a remainder by

$$\varphi(x)=x^2+px+q=0.$$

Now divide $f(x)$ by $\varphi(x)$ and let the result be

$$\frac{f(x)}{\varphi(x)}=\psi(x)+\frac{F_1(p,\,q)x+F_2(p,\,q)}{\varphi(x)}$$

where

$$\psi(x)=x^3+A_1x^2+A_2x+A_3=0,$$

A_1, A_2 and A_3 being functions of p, q, a_1, a_2, and a_3, and where F_1 and F_2 arranged according to q are

$$F_1=q^2(a_1-2p)+q(-a_3+a_2p-a_1p^2+p^3)+a_5=0 \quad \ldots\ldots\ldots (II)$$

and

$$F_2=q^2+q(-a_2+2a_1p-3p^2)+a_4-a_3p+a_2p^2-a_1p^3+p^4=0 \ldots . (III).$$

Now if (II) and (III) were considered simultaneous and solved for p and q, they would give all the ten values each of p and q which would make $\varphi=0$ divide $f(x)=0$ without a remainder, but this would involve the solution of an equation of the tenth degree.

If, however, (II) and (III) be considered identical and the absolute terms and the coefficients of q in the two be equated after dividing (II) by a_1-2p, there results two equations in p, which must possess at least one common root.

These two equations in p are

$$5p^3-6a_1p^2+(2a_1^2+a_2)p+a_3-a_1a_2=0 \ldots\ldots\ldots\ldots (IV),$$

and

$$2p^5-3a_1p^4+(2a_2+a_1^2)p^3-(2a_3+a_1a_2)p^2+(2a_4+a_1a_3)p+a_5-a_1a_4=0 \ldots (V).$$

These equations (IV) and (V) will have a common root when their resultant, R, formed by Sylvester's method, is equal to zero, that is, when

$$R=$$

$$\begin{vmatrix}
5 & -6a_1 & 2a_1^2+a_2 & a_3-a_1a_2 & 0 & 0 & 0 & 0 \\
0 & 5 & -6a_1 & 2a_1^2+a^2 & a_3-a_1a_2 & 0 & 0 & 0 \\
0 & 0 & 5 & -6a_1 & 2a_1^2+a_2 & a_3-a_1a_2 & 0 & 0 \\
0 & 0 & 0 & 5 & -6a_1 & 2a_1^2+a_2 & a_3-a_1a_2 & 0 \\
0 & 0 & 0 & 0 & 5 & -6a_1 & 2a_1^2+a_2 & a_3-a_1a_2 \\
2 & -3a_1 & a_1^2+2a_2 & -2a_3-a_2a_1 & 2a_4+a_1a_3 & a_5-a_1a_4 & 0 & 0 \\
0 & 2 & -3a_1 & a_1^2+2a_2 & -2a_3-a_1a_2 & 2a_4+a_1a_3 & a_5-a_1a_4 & 0 \\
0 & 0 & 2 & -3a_1 & a_1^2+2a_2 & -2a_3-a_1a_2 & 2a_4+a_1a_3 & a_5-a_1a_4
\end{vmatrix}$$

$$=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (VI).$$

This is therefore the condition that the general quintic be solvable by considering (II) and (III) identical.

The value of the common root p is easily found by differentiating R, and this value of p substituted in (II) or (III) gives in general two values of q corresponding. The value of p common to (IV) and (V) with either of the corresponding values of q substituted in $\varphi=0=x^2+px+q$ gives two roots of the proposed equation (I). Two more roots are given by taking the same value of p with the

other corresponding value of q.

We see that the condition $R=0$ among the coefficients requires that, for some one value of p, there shall be two values of q, *i. e.* that, for example, there exist among the roots such a relation as

$$x_1+x_3=x_2+x_4$$

with

$$x_1x_3=x_2x_4,$$

where x_0, x_1, x_2, x_3, x_4 represent the five roots. The other root, the fifth, is then given by $x_0=2p-a_1$.

In case of equal roots, so that for a known value of p the two values of q are equal, or in any case when it is desirable or easier, three of the roots can be found from $\psi(x)=0$ when any pair of corresponding values of p and q are known.

In this discussion no restriction is placed upon the character of the values of a_1, a_2, ... a_5 or p or q. They may be complex.

Whenever, therefore, the coefficients of a quintic satisfy the condition (VI), then p can have at most only nine values (instead of ten as in the general case) and the equation is solvable by this method. q may at the same time have ten values.

Example: Let the given equation be

$$x^5-10x^4+17x^3+76x^2-228x+144=0.$$

In this case the condition (VI) is satisfied and equations (IV) and (V) have a single common root, $p=-3$. Then from (II) or (III), we have

$$q^2+16q-36=0,$$

from which $q=2$ or -18. Therefore

$$x^2-3x+2=0, \text{ from which } x=1 \text{ or } 2.$$

$$\text{or } x^2-3x-18=0, \text{ from which } x=-3 \text{ or } 6,$$

and finally $x=2p-a_1=-6+10=4$, and the roots are 1, 2, -3, 4 and 6.

Urbana, Ill., December 27, 1897.

NOTE ON THE PRACTICAL APPLICATION OF A SUBSTITUTION GROUP IN SPHERICAL TRIGONOMETRY.

By DR. G. A. MILLER.

Several years ago Professor Study published a very valuable article* on spherical trigonometry in which he demonstrated the fertility of the group concept in this branch of elementary mathematics. His efforts were directed toward making the theory more general and more complete, and he explicitly states that the practical geometer would not find any results in his article which he could directly employ.

In what follows, our aim, on the contrary, is entirely practical. It is our belief that a fair knowledge of spherical trigonometry can be obtained more quickly by devoting one lesson to the substitution group given below than would otherwise be possible.

We let a, b, c represent the three sides of a spherical triangle, and α, β, γ the supplements of the angles opposite these sides, taken in order. From the polar triangle we have that the substitution

$$a\alpha.b\beta.c\gamma \dagger$$

will transform any formula into one which is equally true. It is also evident that any formula which has been derived without assigning any particular value to any one of the six parts is transformed into a formula which is equally true by means of the substitutions

$$ab.\alpha\beta \qquad bc.\beta\gamma.$$

Hence such a formula is transformed into one that is equally true if these substitutions are applied successively. In other words, any substitution of the group generated by the given three substitutions transforms any such formula into one that is equally true. This substitution group (G) is well known. It is composed of the following substitutions :

1	
$abc.\alpha\beta\gamma$	$a\alpha.b\beta.c\gamma$
$acb.\alpha\gamma\beta$	$a\beta cab\gamma$
$ab.\alpha\beta$	$a\gamma ba c\beta$
$ac.\alpha\gamma$	$a\beta.ba.c\gamma$
$bc.\beta\gamma$	$a\gamma.b\beta.ca$
	$aa.b\gamma.c\beta$

If a general formula like

*Study, *Saechsischen Abhandlungen*, 20, pp. 87—231.

†This means that a is replaced by α and that α is replaced by a, etc.

$$\sin a\cos\beta+\cos b\sin c+\sin b\cos c\cos\alpha=0$$

is not transformed into itself by any one of the substitutions of G besides unity, we obtain 12 and only 12 formulae from it by transforming it by the substitutions of G. From each one of these 12 formulae we may obtain the other 11 by applying to it the substitutions of G which differ from unity. These 12 formulae thus form a closed system in regard to G. It is of interest to observe the closed systems in regard to the different subgroups of G and to interpret them geometrically. When a general formula like

$$\cos a=\cos b\cos c-\sin b\sin c\cos\alpha$$

is transformed into itself by one substitution of G which differs from unity $(bc.\beta\gamma)$ we may obtain 6 and only 6 formulae from it by transforming it by all the substitutions of G, and from each one of these 6 we may obtain the other 5 in a similar manner. Each of these six formulae may be obtained from any given one of them by transforming it by means of either one of two substitutions of G.

In general, if such a formula is transformed into itself by just k substitutions of G we may obtain $12\div k$ formulae from it by transforming it by the substitutions of G, and all of these can be obtained from any one of them by transforming it by means of the substitutions of G, k substitutions giving the same formula in each case. It is clear that k must be a divisor of 12; *i. e.* the number of the substitutions of G that transform any general formula of the spherical triangle into itself is a divisor of 12.

If we replace α, β, γ by A, B, C respectively, the first half of the substitutions of G form a subgroup which may be employed in a similar manner to derive the formulae of the plane oblique triangle from each other. Hence this group might be made a very useful and general aid to remember the formulae of trigonometry. The explicit introduction of G into the text-books on spherical trigonometry would also furnish the means of making the student thoroughly acquainted with some of the elements of a concept of great fertility.

Cornell University, March, 1898.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

87. Proposed by E. W. MORRELL, A. M., Professor of Mathematics, Montpelier Seminary, Montpelier, Vt.

A and B set out from the same place, and in the same direction. A travels uniform-

ly 18 miles per day, and after 9 days turns and goes back as far as B has traveled during those 9 days; he then turns again, and, pursuing his journey, overtakes B $22\frac{1}{2}$ days after the time they first set out. It is required to find the rate at which B uniformly traveled. [From *Greenleaf's Arithmetic.*]

Solution by J. F. TRAVIS, Student in the Ohio State University, Columbus, Ohio.

Let $\frac{2}{3}$=number of miles B traveled per day. Then

$22\frac{1}{2}\times\frac{2}{3}$=total distance B traveled.

$18\times 9=162$=number of miles traveled by A in 9 days, and

$162-9\times\frac{2}{3}$=number of miles A is from starting point.

$(9\times\frac{2}{3})\div 18=\frac{2}{3}\div 2$=number of days A traveled backwards.

$\therefore\ 9+\frac{2}{3}\div 2$=total number of days A traveled.

$22\frac{1}{2}-(9+\frac{2}{3}\div 2)=13\frac{1}{2}-\frac{2}{3}\div 2$=number of days in which A must overtake B. To overtake B, A must travel $[22\frac{1}{2}\times\frac{2}{3}-(162-9\times\frac{2}{3})]\div 18$ days.

$\therefore\ [22\frac{1}{2}\times\frac{2}{3}-(162-9\times\frac{2}{3})]\div 18=13\frac{1}{2}-\frac{2}{3}\div 2$, from which we find $\frac{2}{3}=10=$ number of days.

This problem was also solved by *F. R. HONEY*, *G. B. M. ZERR*, and *M. A. GRUBER*. Mr. Gruber gave an alegebraic solution and discussed the problem in general.

88. Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics, Curry University, Pittsburg, Pa.

Find the principal of a note given March 19, 1891, bearing interest 6%. Payments: September 1, 1892, \$243.50; January 19, 1893, \$6.90; April 13, 1894, \$19.10; September 19, 1894, \$110.90. Amount due February 22, 1897, \$229.10.

Solution by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.; P. S. BERG, B. Sc., Superintendent of Schools, Larimore, N. D., and NELSON L. RORAY, Professor of Mathematics, South Jersey Institute, Bridgeton, N. J.

The amount \$229.10 due February 22, 1897, has been running 2 years, 5 months, 3 days. Hence the principal is easily found to be \$200.

The payments of \$6.90 and \$19.10 are evidently less than the interest.

Hence the \$200 has been running since September 1st, 1892, or 2 years, 18 days. But the payments \$6.90, \$19.10 and \$110.90 must be added to the \$200, making \$336.90.

Working back we find that this on September 1st, 1892, was \$300. But \$243.50 was paid on this date; hence we have \$543.50 as principal and interest which has been running since March 19, 1891, or 1 year, 5 months, 12 days.

Whence amount=\$543.50. Rate=6%. Time=1 year, 5 months, 12 days, and hence principal is easily found to be \$500.

Also solved by *G. B. M. ZERR*.

89. Proposed by NELSON L. RORAY, South Jersey Institute, Bridgeton, N. J.

Solve by pure arithmetic. A criminal having escaped from prison traveled 10 hours before his escape was known; he was then pursued so as to be gained upon 3 miles an hour: after his pursuers had traveled 8 hours they met an express going at same rate as themselves, who had met the criminal 2 hours and 24 minutes before; in what time from the commencement of the pursuit will they overtake him?

Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.; MARTIN SPINX, Wilmington, O.; F. R. HONEY, Ph. B., New Haven, Conn.; ALBERT J. GIBBS, Salida, Col.; and AMELIA BACH, Salida, Col.

When the pursuers met the express they had been in pursuit 8 hours. When the express met the criminal, the pursuers had been following the criminal $8-2\frac{2}{5}=5\frac{3}{5}$ hours, and the criminal had been escaping for $10+5\frac{3}{5}=15\frac{3}{5}$ hours.

As the express and the pursuers traveled at the same rate, the distance traveled by the criminal in $15\frac{3}{5}$ hours was traveled by the pursuers in $8+2\frac{2}{5}=10\frac{2}{5}$ hours. The pursuers, in this time, gained $10\frac{2}{5}\times 3=31\frac{1}{5}$ miles. This distance was evidently traveled by the criminal in $15\frac{3}{5}-10\frac{2}{5}=5\frac{1}{5}$ hours.

$\therefore$ The criminal's rate of travel was $31\frac{1}{5}\div 5\frac{1}{5}=6$ miles per hour.

The crimiual therefore had the start of $10\times 6=60$ miles.

But the pursuers gained 3 miles per hour. Then, to gain 1 mile they had to travel $\frac{1}{3}$ hour, and to gain the 60 miles they had to travel $60\times\frac{1}{3}=20$ hours= the time required.

Also solved by *J. H. DRUMMOND, WILL RYAN, D. G. DORRANCE, Jr., W. H. DRANE, G. B. M. ZERR, FREMONT CRANE, M. E. GRABER, B. F. YANNEY*, and *J. A. MOORE*.

ALGEBRA.

81. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

Show that

$$\frac{a_1{}^r}{(a_1-a_2)(a_1-a_3)(a_1-a_4)\ldots(a_1-a_n)}$$

$$+\frac{a_2{}^r}{(a_2-a_1)(a_2-a_3)\ldots(a_2-a_n)}+\ldots\ldots\frac{a_n{}^r}{(a_n-a_1)(a_n-a_2)\ldots(a_n-a_{n-1})}$$

is zero if r is less than $n-1$; to 1 if $r=n-1$, and to $a_1+a_2+a_3+\ldots a_n$ if $r=n$.

[*C. Smith's Treatise on Algebra*, Ex. 53, page 104.]

Solution by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, University, Miss.

The fractions being reduced to their least common denominator, every term of the numerator contains the factor a_1-a_2 except the first and the second. If, in the numerator, we put $a_1=a_2$, the first two terms become the same with opposite signs and each of the remaining terms has a zero-factor. Hence the numerator vanishes under this supposition, and, therefore, a_1-a_2 is a factor of it. Similarly every factor of the denominator may be shown to be a factor of the numerator. Now the latter is a homogeneous expression of a degree less than that of the denominator by $n-1-r$, there being $n-1$ factors in the denominator of each of the original fractions.

If $r<n-1$, the numerator is of lower degree than the denominator. But, as proved above, there are as many conditions that cause the numerator to vanish as there are factors in the denominator. In this case the number of these is greater than the degree of the numerator, which is, therefore, identically equal to zero.

If $r=n-1$, the numerator and the denominator are of equal degree, and, being composed of the same factors, the fraction equals 1.

If $r=n$, the degree of the numerator is one greater than that of the denominator. Hence, besides the factors common to both, there must be in the numerator one other factor of the first degree. Since this factor must be symmetrical with reference to a_1, a_2, a_3, etc., it is $a_1+a_2+a_3+\ldots a_n$.

This last is, therefore, the value of the fraction, the numerical coefficient independent of a_1, a_2, a_3, etc., evidently being unity.

Also solved by *C. W. M. BLACK.*

82. Proposed by **B. F. YANNEY, A. M.**, Mount Union College, Alliance, Ohio, and **F. P. MATZ, D. Sc., Ph.D.** Mechanicsburg, Pa.

$$\left.\begin{aligned} y^2+yz+z^2&=a^2\\ z^2+zx+x^2&=b^2\\ x^2+xy+y^2&=c^2 \end{aligned}\right\}\ \text{find } x,\ y,\ \text{and } z.$$

[*C. Smith's Treatise on Algebra*, Ex. 31, page 172.]

I. Solution by **P. S. BERG, A. M.**, Principal of Schools, Larimore, N. D.

From (1)+(2)+(3), $2(y^2+z^2+x^2)+yz+zx+xy=a^2+b^2+c^2$ .....(4).

Squaring (4)

$4(y^2+z^2+x^2)^2+4(y^2+z^2+x^2)(yz+zx+xy)+(yz+zx+xy)^2=(a^2+b^2+c^2)^2$ ..(5).

From $2(1)^2+2(2)^2+2(3)$

$4(y^2+z^2+x^2)^2+4(y^2+z^2+x^2)(yz+zx+xy)-2(yz+zx+xy)^2=2(a^4+b^4+c^2)$..(6).

From $\sqrt{\{[(5)-(6)]/3\}}$

$yz+zx+xy=\pm\sqrt{\{\frac{1}{3}[(2a^2b^2+2b^2c^2+2a^2c^2)-(a^4+b^4+c^4)]\}}$ ....(7).

Put second member$=m$; then from $\sqrt{[6(7)+2(4)]}$

$2(y+z+x)=\pm\sqrt{[2(a^2+b^2+c^2)+6m]}$ ....(8).

From (4)+(7)−2(2) $2y(y+z+x)=a^2-b^2+c^2+m$. .....(9).

From (9)÷(8) and restoring m

$$y=\frac{a^2-b^2+c^2\pm\frac{1}{3}\sqrt{[12a^2b^2-3(a^2+b^2-c^2)^2]}}{\pm\sqrt{\{2(a^2+b^2+c^2)\pm2\sqrt{[12a^2b^2-3(a^2+b^2-c^2)^2]}\}}}.$$

Similarly, $$z=\frac{a^2+b^2-c^2\pm\frac{1}{3}\sqrt{[12a^2b^2-3(a^2+b^2-c^2)^2]}}{\pm\sqrt{\{2(a^2+b^2+c^2)\pm2\sqrt{[12a^2b^2-3(a^2+b^2+c^2)^2]}\}}},$$

and $$x=\frac{b^2+c^2-a^2\pm\frac{1}{3}\sqrt{[12a^2b^2-3(a^2+b^4-c^2)^2]}}{\pm\sqrt{\{2(a^2+b^2+c^2)\pm2\sqrt{[12a^2b^2-3(a^2+b^2-c^2)]}\}}}.$$

II. Solution by **J. SCHEFFER, A. M.**, Hagerstown, Md.

Subtracting (1) from (2) we get $(x-y)(x+y+z)=b^2-a^2$, or putting $x+y+z=s$, $(x-y)s=b^2-a^2$ .....(4), and subtracting (1) from (3), we thus get $(x-z)s=c^2-a^2$ ....(5). From (4) and (5) we obtain $y=(sx+a^2-b^2)/s$ ....(6), and $z=(sx+a^2-c^2)/s$ .....(7). Adding x to both members of (6) and (7), we have $x+y+z=(3sx+2a^2-b^2-c^2)/s$, or $s^2=3sx+2a^2-b^2-c^2$ .....(8), whence

$x=(s^2+b^2+c^2-2a^2)/3s$.....(9), and then by (6) and (7), $y=(s^2+a^2-2b^2+c^2)/3s$.....(10); $z=(s^2+a^2+b^2-2c^2)/3s$.....(11).

Substituting in (3), expanding, and arranging, we obtain,

$s^4-(a^2+b^2+c^2)s^2=a^2b^2+a^2c^2+b^2c^2-a^4-b^4-c^4$ ....(12),

whence $s^2=\frac{1}{2}[a^2+b^2+c^2\pm\sqrt{3}.\sqrt{(2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4)}]$, or putting the irrational part$=S$, so that $S=\sqrt{3}.\sqrt{(2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4)}$ .....(13). $s^2=\frac{1}{2}(a^2+b^2+c^2\pm S)$.....(14).

Substituting now in (9), (10), (11), we obtain

$$x=[3(b^2+c^2-a^2)\pm S]/6s;$$
$$y=[3(a^2+c^2-b^2)\pm S]/6s;$$
$$z=[3(a^2+b^2-c^2)\pm S]/6s.$$

(13), (14), and (15) suffice to determine the values of x, y, and z.

III. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.

(1)−(2) gives $(y-x)(x+y+z)=a^2-b^2$.....(4).

(1)−(3) gives $(z-x)(x+y+z)=a^2-c^2$ .....(5).

(4)÷(5) gives $y=[(b^2-c^2)x+(a^2-b^2)z]/[a^2-c^2]$.....(6).

(6) in (3) gives $(a^4+b^4+3c^4+a^2b^2-3a^2c^2-3b^2c^2)x^2+(a^4+a^2b^2+3b^2c^2-2b^4-3a^2c^2)xz+(a^4-2a^2b^2+b^4)z^2=a^4c^2-2a^2c^4+c^6$.....(7).

Let $z=vx$ in (2) and (7), and divide (7) by (2),

$$(a^4b^2-2a^2b^4+b^6-a^4c^2+2a^2c^4-c^6)v^2+(a^4b^2-2b^6+3b^4c^2+a^2b^4-3a^2b^2c^2-a^4c^2+2a^2c^4-c^6)v=a^4c^2+2a^2c^4-c^6-a^4b^2-b^6-3b^2c^4-a^2b^4+3a^2b^2c^2-3b^4c^2,$$

or $v^2+2Av=B$, suppose. $\therefore v=\pm\sqrt{(A^2+B)}-A$.

$$x=\frac{b}{\sqrt{(1+v+v^2)}}=\frac{b^2+c^2-a^2\pm\frac{1}{3}\sqrt{[12a^2b^2-3(a^4+b^2-c^2)^2]}}{\sqrt{\{2(a^2+b^2+c^2)\pm2\sqrt{[12a^2b^2-3(a^2+b^2-c^2)^2]}\}}}.$$

$\therefore x=(b^2+c^2-a^2\pm C)/D$, suppose.

$y=(a^2+c^2-b^2\pm C)/D$.

$z=(a^2+b^2-c^2\pm C)/D$.

Prof. Cooper D. Schmitt remarks that nine different solutions of this problem can be found in the *Mathematical Magazine*, Vol. II, Nos. 8 and 10, on pages 141 and 193.

Prof. Zerr sent three solutions.

GEOMETRY.

85. Proposed by S. F. NORRIS, Professor of Astronomy and Mathematics, Baltimore City College, Baltimore, Md.

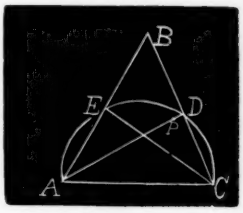
Prove by pure geometry. Give direct proof, if possible.

If the bisectors of two angles of a triangle are equal, the triangle is isosceles.

[From *Wentworth's Plane Geometry*, exercise 43, page 72.]

I. Solution by J. M. COLAW, A. M., Monterey, Va., and EDMUND FISH, Hillsboro, Ill.

In triangle ABC, let AD, CE, be the bisectors of angles A and C, and $AD=CE$. Then angle A=angle C, and triangle ABC is isosceles.



Suppose a circle passed through A, C, and E. It will also pass through D. If not, suppose that it cut AD in any point P short of D. Then arc $EB>$ arc PE, since $\angle ECA(=\angle DCE)>\angle PCE$.

Also, arc PE=arc PC, since $\angle EAP=\angle PAC$. Whence arc $AEP>$arc EPC.

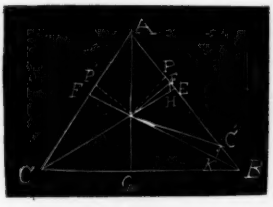
$\therefore$ chord $BP>$chord CE. But by hypothesis, $AD=CE$. $\therefore$ $AP>AD$, which is absurd. In the same way it may be shown that the supposition that the circle, which passes through A, C, and E, cuts BD in any point P', beyond D, also leads to an absurdity. The circle must therefore pass through D. Hence $\angle EAD(=\frac{1}{2}\angle A)=\angle DCE(=\frac{1}{2}\angle C)$.

$\therefore$ $\angle A=\angle C$, and triangle ABC is isosceles.

II. Solution by OTTO CLAYTON, Teacher of Mathematics and Physics, Remington High School, Remington, Ind.

Draw the bisector AG meeting the given bisectors CE and BF of the given triangle ABC, in the point O. Revolve ACG about AG as an axis until AC coincides with AB. Then the point C will fall within the segment AB, on the point B, or without the segment AB, according as $\angle ACO$ is greater, equal to, or less than $\angle ABO$. And the point F will fall within AE, on the point E or within EB, according as $\angle ACO$ is greater, equal to, or less than $\angle ABO$.

But $\angle ACO$ cannot be greater than ABO, for $(C'O+OE)=CE$ would be less than $(BO+OF')=BF$, which is contrary to the hypothesis that $CE=BF$.



Likewise $\angle ACO$ can not be less than $\angle ABO$, for $(B'O+OE)=CE$ would be greater than $(BO+OF)=BF$, which is contrary to the hypothesis. Therefore $\angle ACO=\angle ABO$, and C falls upon B. Therefore the triangle is isosceles.

I think this is a simple proof and does not involve anything outside of the first book of Wentworth.

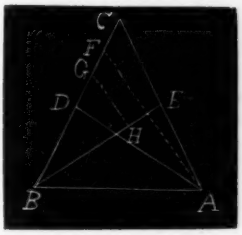
In the proof I did not show how $OE+OC'$ is less than $BO+OF'$. When the perpendicular OP falls between OE and OB the reason is obvious. When OP falls without, construct equilateral triangles $F'OH$ and $C'OK$. Then prove HE less than KB.

III. Solution by the PROPOSER.

Let the bisectors BE and AD be equal, prove triangle ABC isosceles.

Three suppositions are possible.

1st, $\angle A > \angle B$; 2nd, $\angle A < \angle B$; 3rd, $\angle A = \angle B$.



First, suppose $\angle A > \angle B$: then $\frac{1}{2}\angle A > \frac{1}{2}\angle B$. Construct $\angle FAD = \angle CBE$. Then in the triangle FAB, $FB > FA$ (greater side opposite greater angle). Lay off on BF a distance BG equal to AF, and draw GH parallel to FA. Then the triangle BGH=triangle FAD ($BG = FA$, by construction, $\angle GBH = \angle FAD$, for the same reason, $\angle BGH = \angle DFA$, exterior-interior angles.)

$\therefore DH = BA$ (homologous sides of equal triangles) *which is absurd*, because $BE = AD$, by hypothesis, and BH is only a part of BE.

Second, in a similar manner it can be shown that $\angle B$ cannot be greater than A; *i. e.* $\angle A$ cannot be less than $\angle B$.

Third, as $\angle A$ can neither be greater nor less than $\angle B$, it must be equal to $\angle B$. $\therefore$ the triangle is isosceles. Q. E. D.

For other demonstrations of this problem, see Vol. II., pages 158, 189—192. EDITOR.

86. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Prove that the four conics which have S for focus and which touch the three sides of each of the triangles ABC, AEF, BFD, CDE, have their latera-recta equal.

Solution by the PROPOSER.

Reciprocate with respect to S; then we have the three altitudes of an equilateral triangle passing through a point, and the circumscribing circles of the four triangles formed by joining the feet of the perpendiculars equal.

The latera recta of the given conics are then equal.

87. Proposed by WALTER HUGH DRANE, A. M., Professor of Mathematics, Jefferson Military Academy, Washington, Miss.

Given any two straight lines in space, AB, CD, which do not intersect. So construct upon one of the lines as base, a triangle, having its vertex in the other line, such that its perimeter shall be a minimum.

No solution of this problem has been received.

88. Proposed by FREDERICK R. HONEY, Ph. B., Instructor in Mathematics in Trinity College, New Haven, Conn.

Prove that the volume of the frustum of a cone is equal to one-sixth of the altitude multiplied by the sum of the areas of the upper base, the lower base, and four times the area of the section midway between the upper and lower bases.

Solution by FREMONT CRANE, Sand Coulee, Mont.; ALOIS F. KOVARIK, Instructor in Mathematics and Physics, Decorah Institute, Decorah, Iowa; G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va., and the PROPOSER.

Let R=radius of the lower base; r=radius of the upper base; ρ=radius

of the base mid way between the upper and lower bases ; and a=altitude of frustum. Then $\rho=\frac{1}{2}(R+r)$. $\therefore 4\rho^2=(R+r)^2$.

Volume of frustum$=\frac{1}{3}\pi a(R^2+r^2+Rr)=\frac{1}{6}\pi a(2R^2+2r^2+2Rr)=\frac{1}{6}\pi a[R^2+r^2+(R+r)^2]=\frac{1}{6}\pi a(R^2+r^2+4\rho^2)$.

The same method applies to frustums of pyramids, and all solids coming under the prismatoid formula as special cases.

Also solved in a more general manner by *B. F. SINE, P. S. BERG, J. SCHEFFER, HARVEY N. DAVIS, W. H. DRANE,* and *CHAS. C. CROSS.*

CALCULUS.

68. Proposed by EDWARD DRAKE ROE, JR., A. M., Associate Professor of Mathematics, Oberlin College, Oberlin, Ohio.

If $a^{x^{\cdot^{\cdot^{\cdot^{x}}}}}$ to r steps be denoted by $\overset{r}{a^x}$, and if $y=\overset{r}{x^x}$, prove that

$$D_x y=x^{\overset{r}{x}+\overset{r-1}{x}+\cdots+\overset{1}{x}}(\log x)^{r-1}(1+\log x)+\sum_{k=2}^{k=r} x^{\overset{r}{x}+\overset{r-1}{x}+\cdots\overset{k-1}{x}-1}(\log x)^{r-k}.$$

Solution by the PROPOSER.

If $y=f_1(x)^{f_2(x)}$, we obtain, by taking the logarithm of both sides of the equation, and differentiating, the formula

$$D_x y=f_1(x)^{f_2(x)}\log f_1(x) D_x f_2(x)+f_1(x)^{f_2(x)-1} f_2(x) D_x f_1(x).$$

If in this $f_1(x)=x$, $f_2(x)=x^x$, that is if $y=x^{x^x}=\overset{2}{x^x}$, we obtain

$$D_x y=x^{\overset{2}{x}+\overset{1}{x}-1}\log x(1+\log x)+x^{\overset{2}{x}+\overset{1}{x}-1},$$

and the formula to be proved is true when $r=2$. It is evidently not true for values of $r<2$. Assume that it is true for all other values of r.

Let $y=x^{\overset{r+1}{x}}=x^{\overset{r}{x}}$. In the above formula put, $f_1(x)=x$, $f_2(x)=\overset{r}{x^x}$, and we get

$$D_x y=x^{\overset{r+1}{x}}\log x D_x \overset{r}{x^x}+x^{\overset{r+1}{x}-1}\overset{r}{x^x},$$

but by this assumption this is

$$D_x y=x^{\overset{r+1}{x}}\log x[x^{\overset{r}{x}+\overset{r-1}{x}+\cdots\overset{1}{x}}(\log x)^{r-1}(1+\log x)]+x^{\overset{r+1}{x}}\log x\sum_{k=2}^{k=r} x^{\overset{r}{x}+\overset{r-1}{x}+\cdots\overset{k-1}{x}-1}(\log x)^{r-k}$$
$$+x^{\overset{r+1}{x}+\overset{r}{x}-1}$$
$$=x^{\overset{r+1}{x}+\overset{r}{x}+\overset{r-1}{x}+\cdots\overset{1}{x}}(\log x)^r(1+\log x)+\sum_{k=2}^{k=r+1} x^{\overset{r+1}{x}+\overset{r}{x}+\cdots\overset{k-1}{x}-1}(\log x)^{r+1-k}.$$

But this expression has the same form with respect to $r+1$, that the assumption had with respect to r, and since the assumption was true for $r=2$, it is also true for all values of r greater than 2, which is what we had to prove.

Erlangen, Bayern, Hauptstrasse 83II, 26 February, 1898.

Also solved by *C. W. M. BLACK, W. W. LANDIS,* and *G. B. M. ZERR.*

69. Proposed by GEORGE LILLEY, Ph. D., LL. D., Professor of Mathematics, University of Oregon, Eugene, Oregon.

An elliptic fence encloses a field whose major and minor axes are $2a$ and $2b$, respectively. The ends of a rope, the length of which is equal to the length of the fence, are fastened outside the fence and at the extremities of the major axis. A horse is tethered by means of a ring which slides freely on the rope. Over how much ground can he feed? What is the length of the outside border? Find these values in square feet and feet, true to six decimal places, when the area of the field is one acre and $a=2b$.

Solution by MELLEN WOODMAN HASKELL, A. M., Ph. C., Associate Professor of Mathematics, University of California, Berkeley, Cal.

The outside border will evidently be a curve parallel to the given ellipse, so that the two curves will have common normals. Let ds denote the element of arc on the ellipse, dS the element of arc on the parallel curve lying between the same normals, $d\phi$ the angle between those normals, dA the element of area bounded by those normals, by ds and dS, and p the length of the rope, then

$$dS=ds+pd\phi, \text{ and } dA=pds+\tfrac{1}{2}p^2d\phi.$$

Integrating around the ellipse, since in this case the perimeter of the ellipse $\int ds$ is equal to p, and also the complete integral $\int d\phi$ is evidently 2π, we have

$$S=p(1+2\pi) \text{ and } A=p^2(1+\pi).$$

A simple calculation then gives

$p=806.693$ feet, perimeter of ellipse;
$S=5875.293$ feet, length of outside border;
$A=2695155$ square feet, included area.

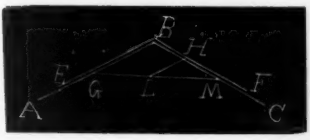
MECHANICS.

57. Proposed by J. C. NAGLE, M. A., M. C. E., Professor of Civil Engineering, Agricultural and Mechanical College of Texas, College Station, Texas.

Over the intersection of two inclined planes slides a cord of uniform mass throughout its length. Find the equation to the path described by its center of gravity.

Solution by the PROPOSER.

Let AB, BC be the inclined planes, EBF the position of the cord at any instant. Let the length of the cord be a; let BC be the x-axis, AB the y-axis. Let $EB=z$. Then the center of gravity of EB will be at G, the mid-point of EB, and of BF at M, the mid-point of BF. Let L be the center of gravity of whole cord. By moments about M,



$$GM\times z=LM\times a, \text{ or } GM/LM=a/z.....(1).$$

But from similar triangles,

$$\frac{GM}{LM}=\frac{BM}{HM}=\frac{\frac{1}{2}(a-z)}{\frac{1}{2}(a-z)-x}=a/z.$$

Whence $2az=z^2+a^2-2ax$ ....(2).

Also from the figure, $GM/LM=\frac{1}{2}z/y=a/z$, from which $z=\sqrt{(2ay)}$.....(3).

Substitute this value of z in (2), square and arrange terms, and there results,

$$4x^2-8xy+4y^2-4ax-4ay+a^2=0 \text{ .....(4).}$$

Equation (4) is that of a parabola, tangent to the axes at a distance of $\frac{1}{2}a$ from the origin.

59. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy in Ohio State University, Athens, Ohio.

Find the radius of a sphere of given specific gravity which will rest just immersed in a fluid whose density varies as its depth.

I. Solution by HENRY HEATON, M. Sc., Atlantic, Iowa; E. L. SHERWOOD, A. M., Superintendent of Schools, West Point, Miss., and W. W. LANDIS, A. M., Professor of Mathematics, Dickinson College, Carlisle, Pa.

Put r=radius of sphere, s_1=its specific gravity, s_2=specific gravity of the fluid at the depth of 1 foot, and a=62.5 pounds=weight of a cubic foot of water. The weight of the sphere is $\frac{4}{3}as_1\pi r^3$. The weight of the sphere must equal the weight of the fluid displaced. To find this, put x=the distance of a horizontal section of the sphere below the surface. The weight of an equivalent quantity of the fluid between two sections whose distance apart is dx is $as_2\pi(r^2-x^2)xdx$. Hence the weight of fluid displaced is

$$as_2\pi\int_0^{2r}(r^2-x^2)xdx=as_2\pi r^4=\tfrac{4}{3}as_1\pi r^3. \quad \therefore\ r=s_1/s_2.$$

II. Solution by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, University, Miss.; and J. C. NAGLE, M. A., M. C. E., Professor of Civil Engineering, in State Agricultural and Mechanical College, College Station, Texas.

Let the origin of coördinates be taken at the lowest point of the sphere, the axis of x vertical, that of y horizontal. The equation of the section of the sphere made by the xy-plane is $y^2=2Rx-x^2$, R being the sphere's radius.

If ρ be the density of the liquid at a depth of unity, then at a distance x above the origin, the density is $\rho(2R-x)$. The weight of the displaced liquid is

$$\int_0^{2R}\pi y^2dx.\rho(2R-x)g.$$

If c be the density of the sphere, its weight is $\frac{4}{3}\pi R^3cg$.

These being equal

$$\tfrac{4}{3}cR^3=\rho\int(2Rx-x^2)(2R-x)dx,$$

the limits being $2R$ and 0, $=\frac{4}{3}\rho R^4$; $R=c/\rho$.

III. Solution by the PROPOSER.

Let δ, δ' be the respective densities of the sphere and of the fluid at a unit depth, and take the upper extremity of the vertical diameter of the sphere for origin, that diameter being in the axis of x.

$\delta' x$=the density of the fluid at depth x, and for equilibrium the mass of sphere immersed equals the mass of fluid displaced.

Then, r being the radius of the sphere,

$$\delta.\tfrac{4}{3}\pi r^3 = \int\int \pi y^2 dx.\delta x = \int_0^{2r} \pi\delta' x(2rx-x^2)dx = \delta'\tfrac{4}{3}\pi r^4. \quad \therefore\ r=\delta/\delta'.$$

Also solved by *J. SCHEFFER* and *G. B. M. ZERR*.

60. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

What must be the ratio of the two legs of a uniform and heavy right triangle suspended from the center of the inscribed circle, if this triangle will rest with the shorter leg in a horizontal position?

I. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

If a and b be the longer and shorter side about the right angles, and r the radius of the inscribed circle, we should have

$$\tfrac{1}{2}r[a+b+\sqrt{(a^2+b^2)}]=\tfrac{1}{2}ab \quad \ldots\ldots(1),$$

$$\text{or } r=\tfrac{1}{2}a+\tfrac{1}{2}b-\tfrac{1}{2}\sqrt{(a^2+b^2)} \quad \ldots\ldots(2).$$

The triangle is kept in equilibrium by the action of the weights of the sides vertically downwards and the reaction of the point of support.

Taking moments about this fallen point,

$$ra=(\tfrac{1}{2}b-r)\sqrt{(a^2+b^2)}+b(\tfrac{1}{2}b-r) \quad \ldots\ldots(3),$$

the sides being taken proportional to their weights.

r from (2) put into (3) gives, by reduction,

$$3a=4b \text{ or } a=\tfrac{4}{3}b \quad \ldots\ldots(4).$$

II. Solution by HENRY HEATON, M. Sc., Atlantic, Iowa, and W.W. LANDIS, A. M., Professor of Mathematics, Dickinson College, Carlisle, Pa.

If a and b represent the legs of the triangle the distance of the center of gravity from the leg a is $\frac{1}{3}b$. The radius of the inscribed circle is

$$r=\tfrac{1}{2}[a+b-\sqrt{(a^2+b^2)}].$$

To secure equilibrium in the desired position r must equal $\frac{1}{3}b$, or

$$\tfrac{1}{3}b=\tfrac{1}{2}[a+b-\sqrt{(a^2+b^2)}].$$

Whence $b=\frac{3}{4}a$, or $b/a=\frac{3}{4}$.

In this triangle the centee of gravity is directly above the center of the inscribed circle. Hence the equilibrium is unstable.

III. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va., and J. SCHEFFER, A. M., Hagerstown, Md.

In order that AB may be horizontal the point of suspension O and the center of gravity G must be in the same straight line perpendicular to AB.

Let $AB=x$, $BC=mx$, $AC=x\sqrt{(1+m^2)}$. Then $HB=r$=radius of in-circle.

$$2(x-r)+2(mx-r)+2r=x+mx+x\sqrt{(1+m^2)}.$$

$\therefore\ r=\frac{1}{2}x[1+m-\sqrt{(1+m^2)}]$.

$GH=\frac{1}{3}BC=\frac{1}{3}mx$, $KH=\frac{1}{2}AB-HB=\frac{1}{2}x-r$.

$\therefore\ KH=\frac{1}{2}x[\sqrt{(1+m^2)}-m)]$, $GK=\frac{1}{3}CK=\frac{1}{3}x\sqrt{(\frac{1}{4}+m^2)}$.

$\therefore\ KG^2=GH^2+KH^2$, or $\frac{1}{9}x^2(\frac{1}{4}+m^2)=\frac{1}{9}m^2x^2+\frac{1}{4}x^2[\sqrt{(1+m^2)}-m]^2$.

$\therefore\ \frac{1}{4}[\sqrt{(1+m^2)}-m]^2=\frac{1}{36}$, or $\sqrt{(1+m^2)}-m=\frac{1}{3}$. $\therefore\ m=\frac{4}{3}$.

$\therefore\ AB : BC=3 : 4$, and $HB : AB : BC : AC=1 : 3 : 4 : 5$.

Also solved by *J. C. NAGLE*.

DIOPHANTINE ANALYSIS.

60. Proposed by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.

It is required to find six positive numbers, such that if each be diminished by five-half times the fifth power of their sum the six remainders will be rational fifth powers.

Solution by the PROPOSER.

Let u, v, w, x, y, z, be the six numbers required, and let $u+v+w+x+y+z=s$.

Then $u-\frac{5}{2}s^5=h^5s^5/q^5$, $v-\frac{5}{2}s^5=k^5s^5/q^5$. $w-\frac{5}{2}s^5=l^5s^5/q^5$,

$x-\frac{5}{2}s^5=m^5s^5/q^5$, $y-\frac{5}{2}s^5=n^5s^5/q^5$, $z-\frac{5}{2}s^5=p^5s^5/q^5$,

Adding these six equations we get

$$s-15s^5=(s^5/q^5)(h^5+k^5+l^5+m^5+n^5+p^5).$$

Let $h^5+k^5+l^5+m^5+n^5+p^5=q^5$. $\therefore\ s=\frac{1}{2}$.

$\therefore\ u=\frac{1}{32}[\frac{5}{2}+(h^5/q^5)]$, $v=\frac{1}{32}[\frac{5}{2}+(k^5/q^5)]$, $w=\frac{1}{32}[\frac{5}{2}+(l^5/q^5)]$,

$x=\frac{1}{32}[\frac{5}{2}+(m^5/q^5)]$, $y=\frac{1}{32}[\frac{5}{2}+(n^5/q^5)]$, $z=\frac{1}{32}[\frac{5}{2}+(p^5/q^5)]$.

Let $h=4$, $k=5$, $l=6$, $m=7$, $n=9$, $p=11$, $q=12$.

$u=\frac{12177}{15552}$, $v=\frac{625205}{796262}\frac{}{4}$, $w=\frac{81}{1024}$, $x=\frac{638887}{796262}\frac{}{7}$, $y=\frac{8803}{32768}$, $z=\frac{7831}{796262}\frac{31}{4}$.

Let $h=5$, $k=10$, $l=11$, $m=16$, $n=19$, $p=29$, $q=30$.

$u=\frac{19441}{248832}$, $v=\frac{12177}{15552}$, $w=\frac{609110}{777600}\frac{51}{000}$, $x=\frac{3862411}{4860000}$, $y=\frac{6322}{777}\frac{6099}{6000}$, $z=\frac{8126}{777}\frac{1149}{6000}$.

AVERAGE AND PROBABILITY.

58. Proposed by HENRY HEATON, M. Sc., Atlantic, Iowa.

From a point on the surface of a circle two lines are drawn to the circumference. Required the average area that may be cut from the circle in this way if the lines are supposed to be drawn at equal angular intervals.

Query I. How does this differ from problem 32?

Query II. Is *sector* the proper word to use for the surface thus cut off?

Query III. Is it absolutely correct to use the word *random* in average problems?

I. Solution by the PROPOSER.

For each pair of lines a second pair may be drawn in opposite directions, dividing the surface of the circle into four portions each of which is included between two of the lines and the circumference. Hence the whole number of surfaces thus cut off may be arranged in sets of four such that the areas of each set shall equal the area of the circle. Hence the average required is $\frac{1}{4}a^2\pi$, where a is the radius of the circle.

Query I. As problem 32 does not describe how the lines are to be drawn to form the "sector" this is a particular case of that problem.

Query II. This query was proposed for information. Some one may be able to give authority for the use of the word in this sense. It is contrary to the usual definition.

Query III. It is the opinion of the writer that the use of the word *random* in average problems is the result of confusion of ideas, and although sometimes convenient is never correct.

II. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.

Let P be the given point. Through P draw the two chords MN, SR dividing the surface of the circle into the four surfaces A, B, C, D.

Then $A+B+C+D=\pi r^2$.

Since P can be taken anywhere on the surface of the circle and the lines MN, SR can make any angle from 0 to π, the average area of A=average area of B=average area of C=average area of D.

$\therefore\ A=B=C=D=\frac{1}{4}\pi r^2$.

After carefully examining problem 32 I am inclined to think the above result the true answer to that problem also.

59. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

A circle is rolling along a horizontal straight line. The uniform velocity of the center is v. Find the average velocity of a point of the circumference.

Solution by JOHN M. COLAW, A. M., Monterey, Va.; JOSIAH H. DRUMMOND, LL. D., Portland, Me.; M. E. GRABER, Mt. Vernon, O.; and the PROPOSER.

For the cycloid traced by the point, we have

$$\left.\begin{aligned} x&=a\theta-a\sin\theta \\ y&=a-a\cos\theta \end{aligned}\right\},$$

$$dx=a(1-\cos\theta)d\theta\ ;\ dy=a\sin\theta d\theta.$$

$\therefore\ ds^2=dx^2+dy^2,=2a^2d\theta^2(1-\cos\theta),=2a^2d\theta^2(2\sin^2\frac{1}{2}\theta).$

$\therefore\ ds=2a\sin\frac{1}{2}\theta d\theta.$

Now $OT=vt=a\theta$. $\therefore\ dt=(a/v)d\theta.$

$\therefore\ ds/dt=2a\sin\frac{1}{2}\theta d\theta\div(a/v)d\theta,=2v\sin\frac{1}{2}\theta$, the variable velocity of P.

$$\therefore \text{ the required average} = \frac{2v\int_0^{\pi} \sin\tfrac{1}{2}\theta d\theta}{\int_0^{\pi} d\theta} = 4v/\pi.$$

Also solved by *G. B. M. ZERR.*

MISCELLANEOUS.

56. Proposed by S. HART WRIGHT, A. M., M. D., Ph. D., Penn Yan, N. Y.

In latitude 40° N.$=\lambda$, when the moon's declination is 5° $23'$ N.$=\delta$, and the sun's declination 9° $52'$ S.$=-\delta'$, how long after sunset will the cusps of the moon's crescent set synchronously, the moon having recently passed its conjunction with the sun?

[NOTE. Problem 56 is identical with problem 54, and need not be here reproduced. See January number, pages 27 and 28, for two solutions. EDITOR.]

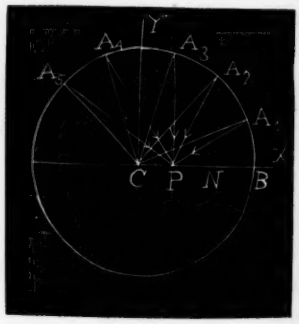
57. Proposed by GEORGE LILLEY, Ph. D., Professor of Mathematics, University of Oregon, Eugene, Ore.

A particle is placed very near the center of a circle, round the circumference of which n equal repulsive forces are symmetrically arranged; each force varies inversely as the mth power of its distance from the particle. Show that the resultant force is approximately $\frac{m_1 n(m-1)}{2r^{m+1}} \times CP$, and tends to the center of the circle, where m_1 is the mass of the particle, CP its distance from the center of the circle, and r the radius of the circle.

I. Solution by the PROPOSER.

Let the particle be at P, and C the center of the circle. Suppose the forces to be at $A_1, A_2, \ldots\ldots$ $CP=x$, and $\angle A_2CB=\theta$. Then $\angle A_1CA_2=\angle A_2CA_3 =\ldots\ldots=360^\circ/n=\beta$, say. Draw A_2N at right angles to CB. Consider the force at A_2. Then,

$$X=[m_1/(A_2P)^m]\cos A_2PN=[m_1(r\cos\theta-x)] /(r^2+x^2-2rx\cos\theta)^{\frac{1}{2}(m+1)}.$$



Let $m_1/r^{\frac{1}{2}(m+1)}=M$. Then, since x is small, neglecting terms containing higher powers of x than the first, we have

$$X=M\left[r^{\frac{1}{2}(1-m)}\cos\theta+\frac{\cos 2\theta\, x\, r^{-\frac{1}{2}(m+1)}}{2} +\tfrac{1}{2}(m-1)xr^{-\frac{1}{2}(m+1)}\right].$$

To obtain the total force on P along X take the sum of n such expressions for all values of θ from $\theta=\alpha$ to $\theta=\alpha+(n-1)\beta$. Hence,

$$\Sigma X = M\left[\frac{r^{\frac{1}{2}(1+m)}\cos[\alpha+\frac{1}{2}(n-1)\beta]\sin\frac{1}{2}n\beta}{\sin\frac{1}{2}\beta}+\frac{\cos[2\alpha+(n-1)\beta]\sin n\beta}{2\sin\beta}xr^{-\frac{1}{2}(m+1)}\right.$$

$$\left.+n(m-1)xr^{-\frac{1}{2}(m+1)}\right]=\frac{m_1 n(m-1)}{2r^{m+1}}x.$$

Similarly, for the force along Y,

$Y=[m_1/(A_2P)^m]\sin A_2PN=Mr\sin\theta(1-2x\cos\theta)^{-\frac{1}{2}(m+1)}=k\sin\theta+k_1x\sin 2\theta$, where $k=Mr^{\frac{1}{2}(1-m)}$ and $k_1=M(m-1)r^{-\frac{1}{2}(m+1)}$.

For the sum of n such expressions, $\Sigma Y=0$. Hence, the resultant is

$$[(\Sigma X)^2+(\Sigma Y)^2]^{\frac{1}{2}}=\frac{m_1 n(m-1)}{2r^{m+1}}\times CP.$$

Since the forces are equal and symmetrically arranged in the circumference, their resultant will act *towards* the center of the circle.

II. Solution by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, University, Miss.

It is assumed that the forces are placed at equal intervals along the circumference, that the repulsive action is proportional to the mass acted on, and that each force acts with unit intensity upon a unit mass at a unit's distance.

Because of the symmetrical arrangement the resultant force will act in the direction PC, C being the center of the circle and P the position of the particle whose mass is m_1.

Let the line connecting the forces with C make angles of α, 2α, 3α, etc., with the radius through P, so that $n\alpha=2\pi$.

The distance of the first force from P is $\sqrt{(r^2-2r.CP.\cos\alpha+CP^2)}$, the cosine of the angle which its direction makes with CP is

$$(r\cos\alpha-CP)/\sqrt{(r^2-2r.CP.\cos\alpha+CP^2)},$$

and the force itself is

$$m_1/(r^2-2r.CP.\cos\alpha+CP^2)^{\frac{1}{2}m}.$$

Writing d for CP, the component of this force along PC is

$$m_1(r\cos\alpha-d)/(r^2-2r\ d\ \cos\alpha+d^2)^{\frac{1}{2}(m+1)}.$$

Putting this in the form

$$m_1(r\cos\alpha-d)(r^2-2r\ d\ \cos\alpha+d^2)^{-\frac{1}{2}(m+1)},$$

expanding by the binomial theorem, multiplying, and neglecting the third and higher powers of d, this becomes,

$$m_1r^{-m-1}\{-d+[r-\tfrac{3}{2}(m+1)r^{-1}d^2]\cos\alpha+(m+1)d\cos^2\alpha$$
$$+\tfrac{1}{2}[m+1)(m+3)]r^{-1}d^2\cos^3\alpha\}.$$

The sum of all such components is

$$m_1 r^{-m-1}\{-nd+[r-\tfrac{3}{2}(m+1)r^{-1}d^2](\cos\alpha+\cos 2\alpha+\cos 3\alpha+\ldots\ldots\cos n\alpha)$$
$$+(m+1)d(\cos^2\alpha+\cos^2 2\alpha+\cos^2 3\alpha+\ldots\ldots\cos n\alpha)$$
$$+\tfrac{1}{2}[(m+1)(m+3)]r^{-1}d^2(\cos^3\alpha+\cos^3 2\alpha+\cos^3 3\alpha+\ldots\ldots\cos^3 n\alpha)\}.$$

By trigonometry,

$$\cos\alpha+\cos 2\alpha+\ldots\ldots\cos n\alpha=\frac{\cos\tfrac{1}{2}(n+1)\alpha.\sin\tfrac{1}{2}n\alpha}{\sin\tfrac{1}{2}\alpha};$$

$$\cos^2\alpha+\cos^2 2\alpha+\ldots\ldots\cos^2 n\alpha=\tfrac{1}{2}\{n+\frac{\cos(n+1)\alpha.\sin n\alpha}{\sin\alpha}\};$$

$$\cos^3\alpha+\cos^3 2\alpha+\ldots\ldots\cos^3 n\alpha=\frac{\cos\tfrac{1}{2}(n+3)\alpha.\sin\tfrac{3}{2}n\alpha}{4\sin\tfrac{3}{2}\alpha}+\frac{3\cos\tfrac{1}{2}(n+1)\alpha.\sin\tfrac{1}{2}n\alpha}{4\sin\tfrac{1}{2}\alpha}.$$

The value of these series when $n\alpha=2\pi$ are 0, $n/2$, and 0, respectively.

Hence the expression for the approximate value of the resultant force reduces to

$$m_1 r^{-m-1}[-nd+(\tfrac{1}{2}n)(m+1)d], \text{ or } \frac{m_1 n(m-1)}{2r^{m+1}}d.$$

Also solved by *G. B. M. ZERR*.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

95. Proposed by WALTER HUGH DRANE, A. M., Professor of Mathematics, Jefferson Military College, Washington, Miss.

Solve by arithmetic, if possible.

A man sold a house for $7500 and gained a certain per cent. on the cost. If the cost had been 16⅔% less, his gain would have been 25% greater. Find the cost of the house.

96. Proposed by RAYMOND SMITH, Tiffin, Ohio.

How many acres in a square field whose diagonal is 10 rods longer than the side?

*** Solutions of these problems should be sent to B. F. Finkel, not later than April 10.

ALGEBRA.

85. Proposed by J. M. COLAW, A. M., Monterey, Va.

Sum the infinite series

$$\frac{1}{1^2.2^2.3^2.4^2.5^2}+\frac{1}{2^2.3^2.4^2.5^2.6^2}+\frac{1}{3^2.4^2.5^2.6^2.7^2}+, \text{ etc.}$$

86. Proposed by J. MARCUS BOORMAN, Consultative Mechanician, Counselor at Law, Inventor, Etc., Woodmere, Long Island, N. Y.

Solve $x^2+yz=16$ .....(A); $y^2+xz=17$.....(B); $z^2+xy=22$.....(C), for all the roots.

[This is Col. Titus' problem—see "Maseres' Tracts," pages 188-276—and is solved by Dr. Wallis in 51 pages, and by Mr. Frend in 38 pages, 8vo., but by the writer in 1 or 2 pages, 4to., or less. J. M. B.]

87. **Proposed by the late B. F. BURLESON, Oneida Castle, N. Y.**

A starts to travel around a circular island at a given point and travels at the rate of 5 miles in 4 hours. One half hour after A, B starts from a point directly opposite from A and travels in an opposite direction at the rate of 4 miles in 3 hours. One hour afterwards C starts from the same point as A and travels in an opposite direction to A at the rate of 3 miles in 2 hours. One half hour afterwards D starts from the same point as B and travels in an opposite direction to B at the rate of 2 miles in 1 hour. Required the size of the island, and when they will all be together, and how far each will have traveled at the accomplishment of this event.

*** Solutions of these problems should be sent to J. M. Colaw, not later than June 10.

GEOMETRY.

94. **Proposed by EDMOND FISH, Hillsboro, Ill.**

A tower $AB=a$, is surmounted by a flag pole $BC=b$. A point D is so taken in a line perpendicular to the foot of the tower that angle BDC is a maximum. Prove that AD is a mean proportional between AC and AB.

95. **Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy in Ohio State University, Athens, Ohio.**

At each point of a parabola is described the rectangular hyperbola of four-pointic contact; prove that the locus of the center of the hyperbola is an equal parabola.

96. **Proposed by W. F. BRADBURRY, A. M., Head Master, Cambridge Latin School, Cambridge, Mass.**

Isosceles triangles are constructed externally on the three sides of a triangle as bases, with the angles at the bases each 30°. The triangle formed by joining the remote vertices (the 120° vertices) of these isosceles triangles is equilateral. [Geometric—not Trigonometric—solution.]

*** Solutions of these problems should be sent to B. F. Finkel, not later than June 10.

CALCULUS.

75. **Proposed by O. W. ANTHONY, M. Sc., Instructor in Mathematics in Boys' High School, New York City.**

Solve the differential equation

$$\frac{d^2y}{dx^2}+n^2y=\frac{6}{x^2}y.$$

76. **Proposed by E. B. ESCOTT, Cambridge, Mass.**

Solve the partial differential equation

$$q^2r+4pqs+p^2t+p^2q^2(rt-s^2)=a^2.$$

[*Forsyth's Differential Equations*, page 376.]

77. **Proposed by T. E. COLE, Columbus, Ohio.**

Derive the equation of a point in the pedal of a bicycle as the wheel rolls along on a plane.

*** Solutions of these problems should be sent to J. M. Colaw, not later than June 10.

MECHANICS.

66. Proposed by B. F. FINKEL, A. M., M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

A conical stick of timber, length a, radius of base r, and density δ, is depressed, apex downward, in a liquid, density δ', so that the base is just level with the liquid. If left free to rise, required the greatest altitude to which it will ascend.

67. Proposed by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.

Find the horizontal and vertical components of the moon's "disturbing force" for any point on the earth's surface making an angle φ with the line joining the center of the earth to the center of the moon.

*** Solutions of these problems should be sent to B. F. Finkel, not later than June 10.

AVERAGE AND PROBABILITY.

63. Proposed by COL. CLARKE.

Three points are taken at random, one on each of the three faces of a tetrahedron; what is the chance that a plane passing through them cuts the fourth edge?

[From *Williamson's Integral Calculus*, page 410.]

64. Proposed by Rev. W. A. WHITWORTH, A. M.

O is a given point within a triangle; P is a random point within the same. The line through O and P is produced so as to divide the triangle into a trapezium and a triangle. Find the average area of this triangle. [From the *Educational Times*, London, Eng.]

*** Solutions of these problems should be sent to B. F. Finkel, not later than June 10.

BOOKS AND PERIODICALS.

Theoretical and Practical Graphics. By Frederick N. Willson, C. E., A. M., Professor in the School of Science, Princeton University. (Author's Edition) 1897. 4to. Pages viii + 264 + Appendix.

This is a most attractive work, not only conquering graphics entire, but containing much more of highest geometric interest, including a fairly complete course on higher plane curves.

The part of the subject where Church so long held supremacy in America with his Descriptive Geometry justly appreciated for its elegance, is paralleled by Professor Willson in his Chapter I and Chapters IX—XII, 117 pages in all, including 219 figures in the text, where he not only covers with equal conciseness and elegance the matter of Church's 138 pages of text and 21 pages of illustrations (102 figures), but in addition has treated many new and important matters, such as the Conoid of Pluecker (articles 333, 356, 477) a favor-

ite surface of Sir Robert Ball, applied in his Theory of Screws, which itself may be looked upon as in part an application of non-Euclidean geometry, also the Cylindroid of Frézier (§§ 333, 360, 489), the corne de vache (§ 361, 475-6), and some special helicoids (§ 480-4), and has also covered the Third Angle (or 'shop') Method of employing descriptive geometry, and given a very full treatment of development (§ 405-20). The mathematical surfaces are beautifully illustrated.

The general plan of the book, while providing a comprehensive graphical training in the form of a progressive course, admits of specialization, of shorter courses, with noticeable flexibility. In fact, eight subgroupings are indicated for independent courses. Comparison with the special treatises scrupulously cited shows the extent of matter on all topics usually treated to be surprisingly great. Professor Willson has a gift for condensing without loss of clearness. With this power, he does well to restate for convenient reference many of the fundamental definitions which he presumes already in some form previously mastered, for example the definition of the trigonometric functions on page 31.

But I still prefer the definition in the note on page 121, "a straight line is the line which is completely determined by two points," to the author's second thought given in the preface, "the line that is completely determined by any two of its points." The spheric space of non-Euclidean geometry, though movable as a whole in itself, is such that two geodetic lines in it always cut in two points. Of course no spherical trigonometry is employed in the author's solution of the problems of trihedrals, pure a graphic process, as it should be. We are glad to find as an appendix the brief but very weighty paper on Trochoids which was so highly and justly praised when presented to the American Association for the Advancement of Science. We cannot forbear to dwell upon the superb illustrations, which make the book a portfolio of art. The author is particularly happy in deciding conflicts of nomenclature, as where he refuses to follow Javary (§ 508) in calling the geodesic on a cone a conical helix.

The author has been extraordinarily painstaking in the proof-reading, and the book is practically free from error. A few trifles have been noticed—

Page 165, § 433, first line, for "prism" read "cylinder."
Page 171, § 442, first line, for "axes" read "bases."
Page 37, sixth line from below, for 90° read 9°.
Page 67, § 194, seventh line, for ϕ read θ.

The slip on page 55, § 166, in stating the brachistochrone and tautochrone properties of the cycloid is so evidently a reference to a reversed or inverted figure inadvertently omitted that it also is trivial.

As to the briefest hint of contents by chapters: I. Definitions, classification. II. Free-hand sketching. III. Draughtsman's outfit. IV. Use of instruments. V. Higher plane curves. VI. Conventional representation. VII. Lettering. The treatment of lettering is particularly full, and sixty-four alphabets are given. VIII. Copying processes. IX. Descriptive geometry of Monge. X. Projections, intersections, development of surfaces, with applications to elbow joints, blast pipes, arch construction, &c. XI. Trihedrals. XII. Projection of sphere. Here the now disused orthographic projection is somewhat condensed, but the stereographic, which is used, is far more complete than in Church. XIII. Shades and shadows. XIV. Perspective. XV. and XVI. Isometric and Clinographic projection, with applications; also, crystals in oblique projection. XVII. Bridge details, toothed gearing, &c. Out of a host of beautiful figures, we may mention 92 as particularly efficient in teaching homology or *complete plane perspective*.

It is a particular pleasure to welcome this book, because it is on just the lines where English and American mathematics has hitherto been sterile.

Even now, the tremendous, the fundamental importance of von Staudt's geometry of position, the pure projective geometry, both for science and philosophy, is realized by few. For example, in the Bolyai type of non-Euclidean geometry, not only is the straight line infinite, but also it has two distinct points at infinity; it is never closed, even by points

at infinity. Writing in 1835, even the superhuman penetration of Lobachévski attributed this essential openness to the straight in itself. In the introduction to his "New Elements of Geometry" he says: "I consider it unnecessary to analyse in detail other assumptions too artificial or arbitrary. Only one of them still deserves some attention; namely, the passing over of the circle into a straight line. Moreover, here the fault is visible from the beginning in the violation of continuity, when a curve, which does not cease to be closed, however great it may be, must change immediately into the most infinite straight line, since in this way it loses an essential characteristic.

In this regard the imaginary geometry (the non-Euclidean geometry) fills out the interval much better. When in it we increase a circle all whose diameters come together at a point; finally, we so attain to a line such that its normals continually approach, although they no longer can cut one another. This characteristic does not pertain to the straight, but to the curve, which, in my paper "On the Foundations of Geometry," I have called *circle-limit*. Of course, it was not until in the next decade (1847) that von Staudt published his immortal "Geometrie der Lage, but long afterward Helmholtz suffers still more seriously for lack of the pure projective geometry, treating the projective questions which necessarily came up in his extended optical researches, sometimes by means and methods of his own make; sometimes only by general reasonings.

Again in *Mind* (1876) Helmholtz misses thus a fundamental difference. He says, p. 315: "It is in fact possible to imagine conditions for bodies apparently solid such that the measurements in Euclid's space become what they would be in spherical or pseudo-spherical space. * * * Think of the image of the world in a convex mirror. * * * Now, Beltrami's representation of pseudo-spherical space in a sphere of Euclid's space is quite similar, except that the background is not a plane, as in the convex mirror, but the surface of a sphere, and that the proportion in which the images as they approach the spherical surface contact, has a different mathematical expression."

But in reality these differences are so fundamental as to make all the difference between Euclidean and non-Euclidean; for the changed measure for distance in the mirror world is still Euclidean, parabolic, using an imaginary conic in the plane background as "absolute" in Cayley's sense. Thus Helmholtz repeated, reproduced the old, but false, theorem that in space of positive curvature two geodetic lines, if they in general cut, must necessarily cut in *two* points. He never attained the conception of single elliptic space, the type-form, but speaks only of "spherical space of three dimensions."

It is to be hoped that Professor Willson's book may hasten the day in America when courses in descriptive geometry and pure projective geometry, no longer confined to science schools, may be available in every college, and when there may be a more adequate realization of the power of spatial imaging as an instrument in scientific research.

Austin, Texas. DR. GEORGE BRUCE HALSTED.

New Psychology. By John P. Gordy, Ph. D., LL. D., Head of the Pedagogical Department of the Ohio State University. 8vo. Cloth, 402 pages. Price, $1.25. New York: Hinds & Noble.

This is the best elementary Psychology that has yet appeared. The plan of this work is the same as that of Dr. Gordy's "Lessons in Psychology," a book published in 1891 and, probably, read by a larger number of teachers than any similar work on the subject. The New Psychology is entirely rewritten, and the subject is treated in a simple, clear and philosophical manner. It should be in the hands of every teacher who desires to acquire a good working knowledge of the development and activity of the human mind. B. F. F.

Mechanical Drawing. By J. C. Tracy, C. E., Instructor in the Sheffield Scientific School of Yale University. 1898. New York: Harper & Brothers.

This book affords an excellent introductory course. Its aim is to prepare the student

for a more extended course in any one of the special lines of drafting. It is comprehensive enough for use in schools and colleges, and, at the same time, is admirably suited to the needs of the student who must study the subject with little, if any, help from a teacher. The book closes with a valuable chapter on Perspective by E. H. Lockwood, M. E., instructor in the Sheffield Scientific School. J. M. C.

Supplemento al Periodico di Mathematica. Diretto dal Dott, Giulio Lazzeri.

Étude sur le Triangle et sur Certains Points de Geometrographie. Par M. E. Lemoine, Ancien élève de l'École Politechnique, á Paris. Pamphlet 24 pages. Reprint from the proceedings of the Edinburgh Mathematical Society, Vol. XIII.

Elementi di Geometria. By G. Lazzeri e A Bassani Professori nella R. Accademia Navale, Livorno, Italia. Large 8vo, paper covers, 380 pages.

This work is divided into five books. Book I contains five chapters. Chapter I treats of lines and planes; chapter II, of segments, angles and dihedrals; chapter III, first notions on circles and spheres; chapter IV, parallel lines and parallel planes; chapter V, lines and planes perpendicular. Book II contains four chapters. Chapter I treats of polygons; chapter II, of solid angles; chapter III, of polyhedria; chapter IV, of distances. Book III contains four chapters. Chapter I treats of the relation between lines, planes and spheres; chapter II, of relations of polygons with circles and of polyhedria with spheres; chapter III, of geometry of the sphere; chapter IV, of surfaces and solids of revolution. Book IV contains seven chapters. Chapter I treats of the general theory of equivalence; chapter II, of the equivalence of polygons and of polyhedric surfaces; chapter III, of the equivalence of spherical polygons and spherical pyramids; chapter IV, of equivalence of prisms; chapter V, of the extent of limits; chapter VI, of the equivalence of polyhedra; chapter VII, of equivalence of circles and the three round bodies—sphere, cylinder and cone. Book V contains five chapters. Chapter I treats of the theory of proportion; chapter II, of homology and similarity; chapter III, of measurement; chapter IV, of the application of algebra to geometry.

From the above outline of the contents of this work, it is clear to be seen that the authors have not followed the old stereotyped order of presenting the subject of geometry, but have treated it in a wholly original manner. The figures of solid geometry are very artistic, representing, as near as it is possible for diagrams to do, the real spatial figure. The book contains 312 figures. At the end of each book there is a long list of theorems for original work, making a total of 1,067. B. F. F.

Mélanges Sur la Géométrie du Triangle. Par M. E. Lemoine, Ancien élève de l'École Politecnique, à Paris.

This paper was presented at the meeting of the French Association for the Advancement of Science, August 8, 1895. In this paper, Mr. Lemoine has derived many interesting properties of the triangle, and established a number of theorems in reference to the triangle and its relation to conic sections. In the Geometry of the Triagle, as studied from the modern point of view, Mr. Lemoine is the recognized leader and authority. B. F. F.

Yale Entrance Examinations in Mathematics. Compiled by Richard Mather, Ph. B. 1898. New Haven, Conn.: Boardman School Press.

This book contains the Yale entrance examinations in methematics from 1884 to 1898, and furnishes a very interesting and useful collection. The book will prove especially acceptable to preparatory and other schools where students are being fitted for college. J. M. C.

Application de la Geometrographic a l'Examen de Diverses Solutions d'un mème Problème. Par M. E. Lemoine, Ancien élève de l'École Politecnique, à Paris. Extract du *Bulletin de la Société Mathematique de France.* Pamphlet, 19 pages.

Questions Relatives a la Géométrie du Triangle, a la Geometrographie et a la Transformation Continue. Par M. E. Lemoine, Ancien éléve de l'École Politecnique, á Paris. Pamphlet, 32 pages.

A paper presented to the French Associatioa for the Advancement of Science, at its meeting April 3, 1896.

Integral Calculus. By Daniel Alexander Murray, Ph. D., Instructor in Mathematics in Cornell University. 288 pages. 1898. New York and Chicago: American Book Company.

This book is one of the Cornell mathematical series. While written primarily for use at Cornell, this text is admirably suited for any one beginning this study. The first two chapters, treating, respectively, of "integration, a process of summation" and "integration, the inverse of differentiation," are more extended than is usual in elementary works, and will prove an important aid to the student in gaining a clear idea of what the Integral Calculus is, and of the uses to which it may be applied. The subject matter is presented in a simple manner, but there has been no sacrifice of rigor of treatment. Chapters of more than ordinary interest are those on Integral Curves and on Ordinary Differential Equations. Many practical problems and illustrative examples are found throughout the book, while the appendix contains important additional matter. In short, the matter, contents and details of treatment characterize a well-constructed text-book, which is worthy the attention of teachers. J. M. C.

The American Journal of Mathematics, January, 1898, has the following leading papers: The Motion of a Solid in Infinite Liquid under no Forces, by A. G. Greenhill; Surfaces of Rotation with Constant Measure of Curvature and their Representation on the Hyperbolic (Cayley's) Plane, by Geo. F. Metzler; Sur les méthodes d'Approximations successives dans la Théorie des Equations différentielles, par Emile Picard. There is a frontispiece portrait of G. Darboux.

The American Monthly Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, $2.50 per year, in advance. Single number, 25 cents. The Review of Reviews Co., New York.

In history-making times like these a truthful record of passing events becomes an imperative need. The daily newspaper is ephemeral and not easily preserved for reference. The *American Monthly Review of Reviews* has all the value of the newspaper, besides distinctive merits of its own. As an epitome of current history it is complete, compact, terse, impartial, absolutely reliable, and judiciously edited. As a piece of journalistic history-writing, what could be more brilliant or fascinating than the May number of this publication, with its story of the Spanish-American war crisis? Merely as a souvenir of this past eventful month the *Review* has a certain unique fitness.

The Cosmopolitan. An International Illustrated Monthly Magazine. Edited by John Brisben Walker. Price, $1.00 per year in advance. Irvington-on-the-Hudson, New York.

One of the most popular literary magazines published in America.

The American Journal of Mathematics, for April, has three interesting papers, as follows: On the Focal Surfaces of the Congruences of Tangents to a given Surface, by A. Pell; Displacements Depending on One, Two and Three Parameters in a Space of Four Dimensions, by Thomas Craig; and, Further Researches in the Theory of Quintic Equations, by Emory McClintock.

The following periodicals have been received since the last acknowledgement: *Journal de Mathématiques Élémentaires*, (1er Mars, 1898); *The American Journal of Mathematics*, (January, 1898); *L' Intermédiaire des Mathématiciens*, (Fevrier 1898); *Miscellaneous Notes and Queries*, (Nov.-Dec., 1897); *Bulletin of the American Mathematical Society*, (February, 1898); *The Kansas University Quarterly*, (January, 1898); *The Monist*, (January, 1898); *The Literary Digest*, (January to March 19, 1898); *The Ohio Teacher*, (March, 1898); *The Educational Times*, (March 1, 1898); *American Journal of Mathematics*, (April, 1898); *L' Intermédiaire des Mathématiciens*, (Mars 1898); *Journal de Mathématiques Élémentaires*, (15 Mars 1898); *The Monist*, (April, 1898); *Bulletin of the American Mathematical Society*, (April, 1898); *The Mathematical Gazette*, (February, 1898); *The Educational Times*, (April, 1898).

NOTES.

In a letter from Dr. Alexander Macfarlane, among other things he says: The following extract from Dr. Laisant's book "La Mathématique : Philosophic Ensciguement" may encourage the editors of the MONTHLY.

Parmi toutes les nations du monde, il u' en est assurément par une plus intérsessante que les Etats-Uuis d' Amerique, au point de vue du développment rapide qu a pris ce pays, des progrés prodigieux de son industrie, de l' énergieet de l' initiative dont il a fait preuve. Mais tant d' activité obligée, imposeé à un pays neuf par les conditions de son existence, se concileait mal avec des recherches pureuant théoriques, avec la poursuite sentimentale de la verité pure. Aussi le nombre des mathématiciens Americains a-t-il été longtemps des plus reduits. Il y a environ vingt-cing aus, mon ami regretté J. Houël l' un des savants les instruits de sou temps, et dont l' érudition mathématique était immense, m' aître tenait dans une de ses lettres de la situation mathématique des diverses nations et arrivait a' cette phrase typique dont je n' ai jamais perdu le souvenir. "Quant aux Etats-Unis, ils importent juste la quantité de science pure que est necessaire à leur industrie."

C' était très' exact alors. Dupuis cette époque, au savant anglais, Sylvester, mort il y a peu de temps, fut appelé a professer a Baltimore ; il y fonda un important journal mathématique, et provoqua une sorte de révolution intellectuelle à ce point de vue, en quelques années seulement. Aujour d' hui, la *Societé Mathematique Americaine* de New York comprend à peu près le même nombre

de membres que la *Societé Mathématique de France*. La Mathématique sons toutes ses formes et dans toutes ses parties, est professée dans une foule de publications et cultiveé par des savants qui ne le cé dent en rien à leurs confréres d' Europe. Elle n' est plus au object d' importation emprunté a l' ancien monde ; c' est devenu un article essential de la production nationale, et cette production augmente chaque join comme importance et comme quantité. Ce phénomène s' est accompli, je le repete, en un trés petit nombre d' années et il est assez curieux pour valoir la peine d' une indication.

Malgré ce développment extraordinaire, et peut-être à cause de ce développment l' industrie Americaine n' a rien perdu de son activité, bien au contraire, elle prend à tâche et parfois avec une sorte de fièvre, de transporter les résultats de la science pure dans le domaine des applications dès qu' elle les juge utilisables ; et c' est par centaines que l' on pourrait compter les publications Americaines s' occupant chaque jour, sous une forme ou sous une autre, de Mathématique appliquée.

Believing this to be of interest to our readers also, we have published the extract in full.

Errata.

Page 27, line 2 from bottom, for "meridian" read *vertical circle.*

Page 43, line 11, for "quadrate" read (*bi-*)*quadrate*; line 18, after "15" insert "Add, etc., $x^2-2xy+y^2=-25$. $\therefore \pm(x-y)=5i$ at once."

Page 44, line 12, after (II), insert "as any figures for (XIII).....(XV) or (*M*) will show"; line 22, for "or" read *and*, and insert (VI) before "above"; line 30, omit "(" before $20\frac{1}{2}$; line 33, for "$x=$" read $x_{||}=$; line 38, for "in" read *into.*

Page 86, line 7, for "$2^{n-3}2(2^2-1)s+t$" read $2^{n-3}[2(2^2-1)s+t]$; line 8, for "$2^{n-4}2(2^3-1)s+t$" read $2^{n-4}[2(2^3-1)s+t]$; line 11, for "$b=x_2=2^{n-1}t$" read $b=x_2=2^{n-2}t$.

Page 89, line 28, after "$8r/\pi$" insert *log* before expression in parenthesis.

Page 90, line 3, omit "to"; line 15, read $\rho=r[1-(4/\pi)\theta]$.

Page 93, line 5 from bottom, for "power" read *powers*, and in next line, for "sum" read *sums.*

Page 95, line 13, for "$\frac{1}{2}P^3$" read $\frac{1}{2}P^5$, and for "S_5" read S_6; line 14, for "....1/42" read ....+1/42 ; in problem 83, read $y^2+x=60$....(2).

Page 96, in problem 73, read $\int_0^{\frac{1}{4}\pi}\log(1+\tan x)dx$; problems 64 and 65 in Diophantine Analysis, should be 66 and 67.

Page 96, problem 92, for $AB\times BC : DC\times AD=BD : AC$, read $AB\times BC+DC\times AD : AB\times AD+BC\times CD=BD : AC$.

The American Journal of Mathematics, for April, has three interesting papers, as follows: On the Focal Surfaces of the Congruences of Tangents to a given Surface, by A. Pell; Displacements Depending on One, Two and Three Parameters in a Space of Four Dimensions, by Thomas Craig; and, Further Researches in the Theory of Quintic Equations, by Emory McClintock.

The following periodicals have been received since the last acknowledgement: *Journal de Mathématiques Élémentaires*, (1er Mars, 1898); *The American Journal of Mathematics*, (January, 1898); *L' Intermédiaire des Mathématiciens*, (Fevrier 1898); *Miscellaneous Notes and Queries*, (Nov.-Dec., 1897); *Bulletin of the American Mathematical Society*, (February, 1898); *The Kansas University Quarterly*, (January, 1898); *The Monist*, (January, 1898); *The Literary Digest*, (January to March 19, 1898); *The Ohio Teacher*, (March, 1898); *The Educational Times*, (March 1, 1898); *American Journal of Mathematics*, (April, 1898); *L' Intermédiaire des Mathématiciens*, (Mars 1898); *Journal de Mathématiques Élémentaires*, (15 Mars 1898); *The Monist*, (April, 1898); *Bulletin of the American Mathematical Society*, (April, 1898); *The Mathematical Gazette*, (February, 1898); *The Educational Times*, (April, 1898).

NOTES.

In a letter from Dr. Alexander Macfarlane, among other things he says: "The following extract from Dr. Laisant's book 'La Mathématique: Philosophic Enseignement' may encourage the editors of the MONTHLY.

Parmi toutes les nations du monde, il n' en est assurément pas une plus intérsessante que les États-Unis d' Amérique, au point de vue du développement rapide qu 'a pris ce pays, dés progrés prodigieux de son industrie, de l' énergie et de l' initiative dont il a fait preuve. Mais tant d' activité obligée, imposeé à un pays neuf par les conditions de son existence, se conciliait mal avec des recherches purement théoriques, avec la poursuite sentimentale de la verité pure. Aussi le nombre des mathématiciens Américains a-t-il été longtemps des plus reduits. Il y a environ vingt-cinq ans, mon ami regretté J. Houël l' un des savants les plus instruits de son temps, et dont l' érudition mathématique était immense, m' entre tenait dans une de ses lettres de la situation mathématique des diverses nations et arrivait à cette phrase typique dont je n' ai jamais perdu le souvenir; "Quant aux États-Unis, ils importent juste la quantité de science pure qui est nécessaire à leur industrie."

C' était très' exact alors. Depuis cette époque, un savant anglais, Sylvester, mort il y a peu de temps, fut appelé à professor à Baltimore; il y fonda un important journal mathématique, et provoqua une sorte de révolution intellectuelle à ce point de vue, en quelques années seulement. Aujour d' hui, la *Société Mathématique Américaine* de New York comprend à peu pres le même nombre

de membres que la *Société Mathématique de France*. La Mathématique sous toutes ses formes et dans toutes ses parties, est professée dans une foule de publications et cultivé par des savants qui ne le cèdent en rien à leurs confréres d' Europe. Elle n' est plus un objet d' importation emprunté à l' ancien monde ; c' est devenu un article essentiel de la production nationale, et cette production augmente chaque jour comme importance et comme quantité. Ce phénomène s' est accompli, je le repète, en un très petit nombre d' années et il est assez curieux pour valoir la peine d' une indication.

Malgré ce développment extraordinaire, et peut-être à cause de ce développement, l' industriè Américaine n' a rien perdu de son activité, bien au contraire, elle prend à tâche, et parfois avec une sorte de fièvre, de transporter les résultats de la science pure dans le domaine des applications des qu' elle les juge utilisables ; et c' est par centaines que l' on pourrait compter les publications Américaines s' occupant chaque jour, sous une forme ou sous une autre, de Mathématique appliquée."

Believing this to be of interest to our readers also, we have published the extract in full.

Errata.

Page 27, line 2 from bottom, for "meridian" read *vertical circle.*

Page 43, line 11, for "quadrate" read (*bi-*)*quadrate*; line 18, after "15" insert "Add, etc., $x^2-2xy+y^2=-25$. $\therefore \pm(x-y)=5i$ at once."

Page 44, line 12, after (II), insert "as any figures for (XIII).....(XV) or (*M*) will show"; line 22, for "or" read *and*, and insert (VI) before "above"; line 30, omit "(" before $20\frac{1}{2}$; line 33, for "$x=$" read $x_{||}=$; line 38, for "in" read *into.*

Page 86, line 7, for "$2^{n-3}2(2^2-1)s+t$" read $2^{n-3}[2(2^2-1)s+t]$; line 8, for "$2^{n-4}2(2^3-1)s+t$" read $2^{n-4}[2(2^3-1)s+t]$; line 11, for "$b=x_2=2^{n-1}t$" read $b=x_2=2^{n-2}t$.

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